Bounds on the Number of Divisors

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Preliminary Lemma

Lemma 1. *Let $d \in \mathbb{N}$ be a divisor of $n \in \mathbb{N}$. Then at least one of d or $k = n/d$ is less than or equal to $\sqrt{n}$.*

Proof 1 (by contradiction). Since d divides n, there exists a natural number k such that $d \cdot k = n$.

Suppose for contradiction that *both* d and k are strictly greater than $\sqrt{n}$:

$$d > \sqrt{n} \quad \text{and} \quad k > \sqrt{n}.$$

Multiplying these two inequalities gives

$$d \cdot k > \sqrt{n} \cdot \sqrt{n} = n,$$

which contradicts $d \cdot k = n$. Therefore, at least one of $d \leq \sqrt{n}$ or $k \leq \sqrt{n}$ must hold. □

Proof 2 (direct). Since d divides n, there exists a natural number $k = n/d$ such that $d \cdot k = n$. We consider two cases.

Case $d \leq \sqrt{n}$. The thesis holds immediately, since d itself is at most $\sqrt{n}$.

Case $d > \sqrt{n}$. We show that $k < \sqrt{n}$:

$$\begin{aligned}
d &> \sqrt{n} && \text{(assumption of this case)} \\
\frac{1}{d} &< \frac{1}{\sqrt{n}} && \text{(taking reciprocals of positive quantities reverses the inequality)} \\
\frac{n}{d} &< \frac{n}{\sqrt{n}} && \text{(multiplying both sides by } n > 0\text{)} \\
k = \frac{n}{d} &< \sqrt{n} && \text{(simplifying } n/\sqrt{n} = \sqrt{n}\text{)}
\end{aligned}$$

In both cases at least one of d or k is less than or equal to $\sqrt{n}$. □

Remark. This lemma tells us that every divisor d of n either is at most $\sqrt{n}$, or its complementary divisor n/d is at most $\sqrt{n}$. In other words, to enumerate *all* divisors of n it suffices to check only the natural numbers up to $\sqrt{n}$.

Main Theorem

Theorem 1. *For every $n \in \mathbb{N}$, the number of divisors $d(n)$ satisfies*

$$d(n) \;\le\; 2\sqrt{n}.$$

Proof. Let D denote the set of all positive divisors of n. By definition $|D| = d(n)$.
Partition of D. We partition D into three pairwise disjoint subsets:

$$\begin{aligned} S_1 &= \left\{\, d \in D \;:\; d < \sqrt{n} \,\right\}, \\ S_2 &= \left\{\, d \in D \;:\; d = \sqrt{n} \,\right\}, \\ S_3 &= \left\{\, d \in D \;:\; d > \sqrt{n} \,\right\}. \end{aligned}$$

Since S_1, S_2, S_3 cover D without overlap:

$$d(n) = |S_1| + |S_2| + |S_3|.$$

Step 1: A bijection between S_1 and S_3.
We claim $|S_1| = |S_3|$. We prove this by exhibiting a bijection

$$\varphi \;:\; S_1 \;\longrightarrow\; S_3, \qquad \varphi(d) = \frac{n}{d}.$$

We verify in turn that φ is well-defined, injective, and surjective.[1]

Well-definedness.
Let $d \in S_1$ be arbitrary; we must show $\varphi(d) = n/d \in S_3$. Since $d \in S_1$, d divides n, so $n/d \in \mathbb{N}$ and n/d divides n. It remains to check $n/d > \sqrt{n}$. Since $d < \sqrt{n}$,

$$\begin{aligned} &\frac{1}{d} > \frac{1}{\sqrt{n}} && \text{(taking reciprocals of positive quantities reverses the inequality)} \\ &\frac{n}{d} > \frac{n}{\sqrt{n}} = \sqrt{n} && \text{(multiplying both sides by } n > 0\text{)} \end{aligned}$$

Hence $\varphi(d) \in S_3$ for all $d \in S_1$, and φ is well-defined.

Injectivity.
Suppose $\varphi(a) = \varphi(b)$ for some $a, b \in S_1$, i.e. $n/a = n/b$.

$$\frac{n}{a} = \frac{n}{b} \;\Longrightarrow\; n = \frac{na}{b} \;\Longrightarrow\; bn = an \;\Longrightarrow\; b = a.$$

Hence φ is injective.

[1] Let $f\colon A \to B$ be a function. f is *injective* if $f(a) = f(b) \Rightarrow a = b$. f is *surjective* if $\forall\, b \in B \;\; \exists\, a \in A$ such that $f(a) = b$.

Surjectivity.

Scratch work.

$$\varphi(a) = b \implies \frac{n}{a} = b \implies n = ab \implies a = \frac{n}{b}.$$

Let $b \in S_3$ be arbitrary. Set $a := n/b$. Since $b \in S_3$, b divides n, so $a \in \mathbb{N}$ and a divides n. Moreover, since $b > \sqrt{n}$,

$$a = \frac{n}{b} < \frac{n}{\sqrt{n}} = \sqrt{n},$$

so $a \in S_1$. Then

$$\varphi(a) = \frac{n}{a} = \frac{n}{\frac{n}{b}} = n \cdot \frac{b}{n} = b.$$

Since $b \in S_3$ was arbitrary, φ is surjective.

Having verified that φ is well-defined, injective, and surjective, we conclude that φ is a bijection from S_1 to S_3, and therefore $|S_1| = |S_3|$.

Step 2: Bounding $|S_1|$.

Using $|S_1| = |S_3|$, we rewrite the divisor count as

$$d(n) = |S_1| + |S_2| + |S_1|.$$

Since S_1 consists of distinct naturals all strictly less than $\sqrt{n}$, its elements are among $\{1, 2, \ldots, m\}$ where $m < \sqrt{n}$ is the largest such natural (if $S_1 = \varnothing$, i.e. $n = 1$, then $|S_1| = 0 < 1 = \sqrt{n}$ trivially); hence

$$|S_1| \leq m < \sqrt{n}.$$

Step 3: Final bounds.

We distinguish two cases.

Case A: n is not a perfect square.

$\sqrt{n} \notin \mathbb{N}$, so no divisor of n can equal $\sqrt{n}$; hence $S_2 = \varnothing$ and $|S_2| = 0$. Therefore

$$d(n) = |S_1| + 0 + |S_1| = 2|S_1| < 2\sqrt{n}.$$

Case B: n is a perfect square.

$\sqrt{n} \in \mathbb{N}$ and $\sqrt{n}$ divides n (since $\sqrt{n} \cdot \sqrt{n} = n$). Thus S_2 contains exactly one element, namely $\sqrt{n}$, so $|S_2| = 1$. Therefore

$$d(n) = |S_1| + 1 + |S_1| = 2|S_1| + 1 < 2\sqrt{n} + 1.$$

Since n is a perfect square, $2\sqrt{n} \in \mathbb{N}$, so $d(n) < 2\sqrt{n} + 1$ tightens to[2]

$$d(n) \leq 2\sqrt{n}.$$

In both cases we obtain $d(n) \leq 2\sqrt{n}$, which completes the proof. $\square$

[2] For any two integers a and b, $a < b+1 \Leftrightarrow a \leq b$, since there is no integer strictly between b and $b+1$. Note that this would fail for reals: $a = b + 0.5$ satisfies $a < b + 1$ but not $a \leq b$.