# There Are Infinitely Many Prime Numbers

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**Theorem 1** (Euclid's theorem)**.** *The set of prime numbers is infinite.*

*Proof.* Suppose for contradiction that there are only finitely many primes, say $k$ in total. List them as $p_1, p_2, p_3, \ldots, p_k$, multiply them all together, and consider the number that follows their product:

$$n = p_1 \times p_2 \times p_3 \times \cdots \times p_k + 1.$$

**Case 1: $n$ is prime.** Then we have found a prime not in the list of all $p_i$ — a contradiction, since the list was supposed to be exhaustive.

**Case 2: $n$ is composite.** By the Fundamental Theorem of Arithmetic, $n$ must be a product of primes. However, none of the primes in the list can be a divisor of $n$, because the remainder of every such division is 1. It follows that the prime factors of $n$ are primes outside the list — again a contradiction.

Since both cases lead to a contradiction, the assumption that there are finitely many primes is false. Therefore, the primes are infinite. □

**Remark 1** (Why the remainder is always 1)**.** *Think of it this way: the product $p_1 \times p_2 \times \cdots \times p_k$ is a multiple of every $p_i$ in the list — each one is literally baked into the product as a factor. A multiple of $p_i$ is, by definition, something that $p_i$ divides exactly, leaving no remainder. Now $n$ is obtained by adding $1$ to that product, stepping exactly one unit past a number that $p_i$ divides cleanly. So no matter which $p_i$ you try to divide by you always get a remainder of $1$.*