Irrationality of $\sqrt{2}$

0xHaru

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Theorem 1. *$\sqrt{2}$ is not rational.*

Proof. Assume, for contradiction, that $\sqrt{2}$ is a rational number, meaning that

$$\exists\, a, b \in \mathbb{Z} : \frac{a}{b} = \sqrt{2}.$$

If a and b have a common factor, it can be eliminated. Then $\sqrt{2}$ can be written as an irreducible fraction $\frac{a}{b}$ such that a and b are *coprime*, i.e. they have no common factor other than 1. This additionally means that at least one of a or b must be odd.

It follows that

$$\frac{a^2}{b^2} = 2 \qquad \text{and therefore} \qquad a^2 = 2b^2.$$

Hence a^2 is even, because it equals $2b^2$, which is necessarily even as it is two times another integer. Since squares of odd integers are never even, it follows that a itself must be even.

Because a is even, there exists an integer k such that $a = 2k$. Substituting into $a^2 = 2b^2$ gives

$$2b^2 = a^2 = (2k)^2 = 4k^2 \implies b^2 = 2k^2.$$

It follows that b^2 is also even, which in turn means that b is even.

So a and b are both even, but this contradicts the assumption that $\frac{a}{b}$ is irreducible, since a and b would share the (non-trivial) common factor 2.

This means that $\sqrt{2}$ is not rational. □

Bonus: Odd Squares

Theorem 2. *Let $n \in \mathbb{Z}$. If n is odd, then n^2 is also odd.*

Proof. Since n is odd, there exists an integer k such that

$$n = 2k + 1.$$

Squaring both sides gives

$$n^2 = (2k+1)^2 = 4k^2 + 2(2k)(1) + 1^2 = 4k^2 + 4k + 1 = 2\big(2k^2 + 2k\big) + 1.$$

Setting $m = 2k^2 + 2k \in \mathbb{Z}$, this simplifies to

$$n^2 = 2m + 1,$$

hence n^2 is odd. $\square$