Peano Arithmetic

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Peano Axioms

We first define the **successor function** $S : \mathbb{N} \to \mathbb{N}$ by $S(n) = n + 1$. For example, $S(0) = 1$.

The four Peano axioms are:

Axiom 1 (A1)**.** 0 is a natural number.

Axiom 2 (A2)**.** For every natural number n, $S(n)$ is a natural number.

Axiom 3 (A3)**.** For all natural numbers n and m, if $S(n) = S(m)$ then $n = m$.

Axiom 4 (A4)**.** There does not exist a natural number n such that $S(n) = 0$.

Addition

Addition on $\mathbb{N}$ is defined by the following two rules, for all $n, m \in \mathbb{N}$:

Definition 1 (D1)**.** $n + 0 = n$

Definition 2 (D2)**.** $n + S(m) = S(n + m)$

Lemma 1: $S(a) = a + 1$

Lemma 1. *For all* $a \in \mathbb{N}$, $S(a) = a + 1$.

Proof.

$$S(a) \underset{\text{D1}}{=} S(a+0) \underset{\text{D2}}{=} a + S(0) \underset{\text{Def. of 1}}{=} a + 1.$$

□

Proof that $2 \neq 1$

Theorem 1. $2 \neq 1$, *i.e.,* $S(S(0)) \neq S(0)$.

Proof. We want to prove $\neg\big(S(S(0)) = S(0)\big)$. Suppose for contradiction that $S(S(0)) = S(0)$ is true. Then:

$$S(S(0)) = S(0) \underset{\text{A3}}{\Longrightarrow} S(0) = 0.$$

But A4 states that there is no natural number n with $S(n) = 0$, so $S(0) = 0$ is impossible. Contradiction. □

Lemma 2: $S(a) + b = S(a + b)$

Lemma 2. *For all* $a, b \in \mathbb{N}$, $S(a) + b = S(a + b)$.

Proof. By induction on b.

Base case $(b = 0)$:

$$\begin{aligned} S(a) + 0 &= S(a + 0) & \\ S(a) &= S(a) & \text{by D1 (twice)} \\ a &= a & \text{by A3.} \end{aligned}$$

Induction Hypothesis (I.H.): $S(a) + b = S(a + b)$.

Induction Step. We show $S(a) + S(b) = S(a + S(b))$:

$$\begin{aligned} S(a) + S(b) &= S\big(a + S(b)\big) & \\ S\big(S(a) + b\big) &= S\big(a + S(b)\big) & \text{by D2} \\ S\big(S(a) + b\big) &= S\big(S(a + b)\big) & \text{by D2} \\ S\big(S(a + b)\big) &= S\big(S(a + b)\big) & \text{by I.H.} \\ a + b &= a + b & \text{by A3 (twice).} \end{aligned}$$

□

Proof of Associativity

Theorem 2. *For all* $a, b, c \in \mathbb{N}$,

$$(a + b) + c = a + (b + c).$$

Proof. By induction on c.

Base case $(c = 0)$:

$$(a + b) + 0 \underset{\text{D1}}{=} a + b \underset{\text{D1}}{=} a + (b + 0).$$

Induction Hypothesis (I.H.): $(a + b) + c = a + (b + c)$.

Induction Step. We show $(a + b) + S(c) = a + (b + S(c))$:

$$\begin{aligned} (a + b) + S(c) &= S\big((a + b) + c\big) & \text{by D2} \\ &= S\big(a + (b + c)\big) & \text{by I.H.} \\ &= a + S(b + c) & \text{by D2} \\ &= a + (b + S(c)) & \text{by D2.} \end{aligned}$$

The I.H. holds for $S(c)$. □

Proof of the Identity Element

Theorem 3. *For all* $a \in \mathbb{N}$, $0 + a = a$.

Proof. D1 states that 0 is a *right* identity ($a + 0 = a$). We prove that 0 is also a *left* identity by induction on a.

Base case ($a = 0$): $0 + 0 = 0$ by D1.

Induction Hypothesis (I.H.): $0 + a = a$.

Induction Step. We show $0 + S(a) = S(a)$:

$$\begin{aligned} 0 + S(a) &= S(0 + a) && \text{by D2} \\ &= S(a) && \text{by I.H.} \end{aligned}$$

□

Proof of Commutativity

Theorem 4. *For all* $a, b \in \mathbb{N}$, $a + b = b + a$.

Proof. By induction on b. We first establish the two base cases $b = 0$ and $b = S(0) = 1$ (i.e. we prove that 0 and 1 commute with everything).

Base case ($b = 0$):

$$a + 0 \underset{\text{D1}}{=} a \underset{\text{Identity}}{=} 0 + a.$$

The second equality is the left-identity theorem proved above.

Base case ($b = 1$): We prove $a + 1 = 1 + a$ by induction on a *(an induction proof within an induction proof)*.

- **Inner base case** ($a = 0$):

$$\begin{aligned} 0 + 1 &= 1 + 0 \\ 1 &= 1 + 0 && \text{by the Identity Element} \\ 1 &= 1 && \text{by D1.} \end{aligned}$$

- **Inner Induction Hypothesis (I.H.):** $a + 1 = 1 + a$.

- **Inner Induction Step.** We show $S(a) + 1 = 1 + S(a)$:

$$\begin{aligned} S(a) + 1 &= S(a) + S(0) && \text{by definition of 1} \\ &= S(S(a) + 0) && \text{by D2} \\ &= S(S(a)) && \text{by D1} \\ &= S(a + 1) && \text{by Lemma 1} \\ &= S(1 + a) && \text{by I.H.} \\ &= 1 + S(a) && \text{by D2.} \end{aligned}$$

We have proved the base case $b = 1$.

Induction Hypothesis (I.H.): $a + b = b + a$.

Induction Step. We show $a + S(b) = S(b) + a$:

$$\begin{aligned}
a + S(b) &= a + (b + 1) && \text{by Lemma 1} \\
&= (a + b) + 1 && \text{by Associativity} \\
&= (b + a) + 1 && \text{by I.H.} \\
&= b + (a + 1) && \text{by Associativity} \\
&= b + (1 + a) && \text{by base case } b = 1 \\
&= (b + 1) + a && \text{by Associativity} \\
&= S(b) + a && \text{by Lemma 1.}
\end{aligned}$$

□