Maximum Number of Segments Connecting the Vertices of a Polygon

0xHaru

April 11, 2026

Problem Statement

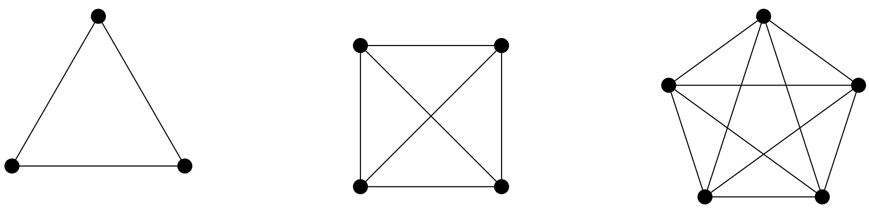
Given a convex polygon with n vertices, how many distinct line segments can be drawn by connecting every pair of vertices?

Setup

A polygon with n sides has exactly n vertices, and initially contributes n segments, namely its *sides*. In addition to the sides, we must also count the *diagonals* — the segments that connect two non-adjacent vertices, passing through the interior of the polygon.

Examples

The figures below show the complete set of segments for polygons with 3, 4, and 5 vertices, respectively.



$n = 3$ $n = 4$ $n = 5$

Counting the Diagonals

Each vertex connects to every other vertex except itself, yielding $n - 1$ connections per vertex. Two of these connections are sides of the polygon, so exactly

$$(n-1) - 2 = n - 3$$

of them are diagonals. Since there are n vertices, a naive count gives $n(n-3)$ diagonals. However, each diagonal is shared by exactly two endpoints, so we have counted every diagonal twice. The correct number of distinct diagonals is therefore:

$$\frac{n(n-3)}{2}$$

Total Number of Segments

Adding the n sides to the diagonals, the total number of segments is:

$$n + \frac{n(n-3)}{2}$$

Connection to Graph Theory

This problem is a classical result in **graph theory**. Modelling the polygon's vertices as the nodes of an undirected graph [1] on n vertices, the question becomes: *what is the maximum number of edges in a simple undirected graph on n vertices?*

The answer is well known:

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

The Handshaking Lemma

An elegant way to arrive at this result is through the **Handshaking Lemma**. Imagine the n vertices as people and each edge as a handshake between a pair of people. We wish to count the total number of handshakes if every pair shakes hands exactly once.

Each person can shake hands with the remaining $n-1$ people, suggesting a total of $n(n-1)$ handshakes. However, this counts every handshake *twice*: Alice shaking Bob's hand and Bob shaking Alice's hand are the same event. Dividing by 2 gives the correct count:

$$\frac{n(n-1)}{2}$$

[1] This graph is both simple and complete: it has no self-loops or repeated edges, and every pair of distinct vertices is connected by a unique edge.

Algebraic Equivalence

We can verify that the formula obtained above, $n + \frac{n(n-3)}{2}$, equals $\frac{n(n-1)}{2}$ by elementary algebra:

$$\begin{aligned} n + \frac{n(n-3)}{2} &= \frac{2n}{2} + \frac{n(n-3)}{2} \\ &= \frac{2n + n(n-3)}{2} \\ &= \frac{n\big(2 + (n-3)\big)}{2} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

The two approaches are therefore perfectly consistent: the geometric argument and the graph argument both yield $\frac{n(n-1)}{2}$ total segments.