# Products of Integers of Equal Parity

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**Definition 1** (Even integer)**.** For $n \in \mathbb{Z}$, $n$ is *even* if and only if

$$\exists\, k \in \mathbb{Z} : n = 2k.$$

**Theorem 1.** *For all $n, m \in \mathbb{Z}$, if $n$ and $m$ are even, then $n \cdot m$ is even.*

*Proof.* Since $n$ and $m$ are even, there exist $i, j \in \mathbb{Z}$ such that

$$n = 2i \quad \text{and} \quad m = 2j.$$

Then

$$n \cdot m = 2i \cdot 2j = 2(2ij).$$

Let $k = 2ij$. Since $i, j \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication, $k \in \mathbb{Z}$. Hence there exists $k \in \mathbb{Z}$ such that $n \cdot m = 2k$, so by definition $n \cdot m$ is even. □

**Definition 2** (Odd integer)**.** For $n \in \mathbb{Z}$, $n$ is *odd* if and only if

$$\exists\, k \in \mathbb{Z} : n = 2k + 1.$$

**Theorem 2.** *For all $n, m \in \mathbb{Z}$, if $n$ and $m$ are odd, then $n \cdot m$ is odd.*

*Proof.* Since $n$ and $m$ are odd, there exist $i, j \in \mathbb{Z}$ such that

$$n = 2i + 1 \quad \text{and} \quad m = 2j + 1.$$

Then

$$n \cdot m = (2i + 1)(2j + 1) = 4ij + 2i + 2j + 1 = 2(2ij + i + j) + 1.$$

Let $k = 2ij + i + j$. Since $i, j \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication and addition, $k \in \mathbb{Z}$. Hence there exists $k \in \mathbb{Z}$ such that $n \cdot m = 2k + 1$, so by definition $n \cdot m$ is odd. □