Why Dividing by a Fraction Means Multiplying by Its Reciprocal

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Property 1 (Invariant Property)**.** Multiplying or dividing both the dividend and the divisor by the same non-zero number leaves the quotient unchanged.

Formally, for all $a, b, k \in \mathbb{Z}$ with $b \neq 0$ and $k \neq 0$:

$$\frac{a}{b} = \frac{a \cdot k}{b \cdot k} = \frac{a/k}{b/k}$$

Definition 1 (Neutral Element)**.** The neutral element of division is 1.

That is, for any $a \in \mathbb{Z}$:

$$\frac{a}{1} = a$$

Theorem 1. Dividing any number by a fraction is equivalent to multiplying that number by the reciprocal of the fraction. That is, for all $a, c, d \in \mathbb{Z}$ with $c \neq 0$ and $d \neq 0$:

$$\frac{a}{\frac{c}{d}} = a \cdot \frac{d}{c}$$

Proof. Let $a, c, d \in \mathbb{Z}$, with $c \neq 0$ and $d \neq 0$, so that $\frac{c}{d}$ is a well-defined, non-zero fraction.

We apply the **Invariant Property of Division**: multiplying both the dividend and the divisor by the same non-zero number does not change the quotient. Here we choose to multiply both by $\frac{d}{c}$, the reciprocal of the divisor $\frac{c}{d}$:

$$\frac{a}{\frac{c}{d}} = \frac{a \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{a \cdot \frac{d}{c}}{1} = a \cdot \frac{d}{c}$$

The key step is that $\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{dc} = 1$, since a fraction multiplied by its reciprocal always equals 1 (the neutral element of division). The result follows immediately. □

Corollary 1. Dividing a fraction by a second fraction is equivalent to multiplying the first by the reciprocal of the second. For all $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

Proof. This follows directly from Theorem 1 by setting $a \leftarrow \frac{a}{b}$. $\square$