The Tower of Hanoi

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April 14, 2026

1 Preliminaries

Definition 1.1 (Sequence)**.** A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence and we use the notation $\{a_n\}$ to refer to the sequence itself.

Definition 1.2 (Recurrence Relation)**.** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence.)

We say that we have solved a recurrence relation when we find an explicit formula, called a **closed formula**, for the terms of the sequence. Such a formula is only meaningful relative to a specific set of **initial conditions**, which specify the terms that precede the first one governed by the recurrence relation. Together, a recurrence relation and its initial conditions uniquely determine the sequence, as one can prove by **mathematical induction**.

2 The Tower of Hanoi

The Tower of Hanoi is a classic puzzle consisting of three pegs — P_1, P_2, P_3 — and a stack of n disks of decreasing size, all initially placed on P_1. The goal is to move the entire stack to P_3, subject to two rules: only one disk may be moved at a time, and no disk may ever be placed on top of a smaller one.

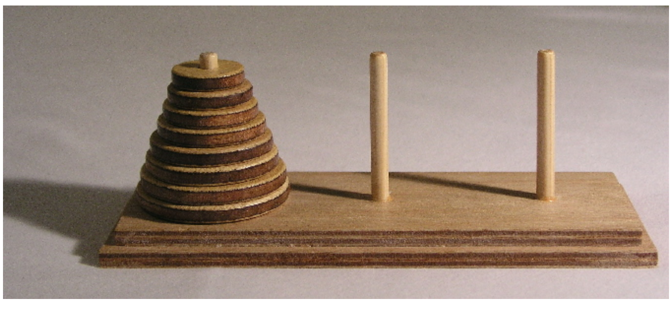

Figure 1: A physical Tower of Hanoi puzzle with all disks stacked on P_1.

3 A Worked Example

We illustrate the puzzle with $n = 3$ disks. Label the disks D_1 (smallest), D_2 (medium), and D_3 (largest), all initially stacked on P_1. The table below shows the move sequence together with the resulting state of every peg after each move, with disks listed from bottom to top. The puzzle is solved in $2^3 - 1 = 7$ moves, confirming the formula we will derive in Section 5.

Move	**Disk moved**	P_1	P_2	P_3
0	(initial state)	D_3, D_2, D_1	–	–
1	$D_1 \to P_3$	D_3, D_2	–	D_1
2	$D_2 \to P_2$	D_3	D_2	D_1
3	$D_1 \to P_2$	D_3	D_2, D_1	–
4	$D_3 \to P_3$	–	D_2, D_1	D_3
5	$D_1 \to P_1$	D_1	D_2	D_3
6	$D_2 \to P_3$	D_1	–	D_3, D_2
7	$D_1 \to P_3$	–	–	D_3, D_2, D_1

Table 1: Solution of the Tower of Hanoi with $n = 3$ disks. Peg contents are listed from bottom to top; – denotes an empty peg.

4 Setting Up the Recurrence Relation

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. We want to set up a recurrence relation for the sequence $\{H_n\}$.

Begin with n disks on P_1.

Step 1. Transfer the top $n-1$ disks from P_1 to P_2 using H_{n-1} moves. The largest disk remains fixed on P_1 throughout.

Step 2. Use 1 move to transfer the largest disk from P_1 to P_3.

Step 3. Transfer the $n-1$ disks from P_2 to P_3 using H_{n-1} additional moves, placing them on top of the largest disk and thus solving the puzzle.

This reasoning shows that

$$H_n = 2H_{n-1} + 1. \tag{1}$$

Initial condition. Since a single disk can be transferred from P_1 to P_3 in exactly one move, we have

$$H_1 = 1.$$

5 Solving the Recurrence Relation

We want to find a *closed-form expression* for (1). We use the **iterative approach**.

5.1 Forward Iteration

Applying the recurrence relation starting from the initial condition:

$$\begin{aligned}
H_1 &= 1, \\
H_2 &= 2H_1 + 1 = 2(1) + 1 = 2 + 1, \\
H_3 &= 2H_2 + 1 = 2(2 + 1) + 1 = 2^2 + 2 + 1, \\
H_4 &= 2H_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1.
\end{aligned}$$

We begin to see the pattern:

$$H_k = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1.$$

Replacing k with the general index n gives the closed form:

$$H_n = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1.$$

5.2 Backward Iteration

We can reach the same conclusion by unrolling H_n directly:

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2\big(2H_{n-2} + 1\big) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2\big(2H_{n-3} + 1\big) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&= 2^3\big(2H_{n-4} + 1\big) + 2^2 + 2 + 1 = 2^4 H_{n-4} + 2^3 + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1.
\end{aligned}$$

The pattern after k substitutions is always

$$H_n = 2^k H_{n-k} + 2^{k-1} + 2^{k-2} + \cdots + 2 + 1.$$

We stop when the subscript reaches 1, i.e. when $n - k = 1$, giving $k = n - 1$. Substituting:

$$H_n = 2^{n-1} H_1 + 2^{n-2} + \cdots + 2 + 1 = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1,$$

where the last equality uses $H_1 = 1$.

We have reached the same conclusion by both approaches. The expression on the right is a *geometric series*.

5.3 Closed Form via the Geometric Series Formula

Recall that

$$a + ar + ar^2 + \cdots + ar^m = \sum_{k=0}^{m} ar^k = \begin{cases} \dfrac{a\,r^{m+1} - a}{r - 1} & \text{if } r \neq 1, \\ (m+1)\,a & \text{if } r = 1. \end{cases}$$

In our case $a = 1$ and $r = 2$, so

$$H_n = \sum_{k=0}^{n-1} 2^k = \frac{2^{n-1+1} - 1}{2 - 1} = 2^n - 1.$$

6 Verification by Induction

Now we want to show that $H_n = 2^n - 1$ is indeed a solution to the recurrence relation $H_n = 2H_{n-1} + 1$ with initial condition $H_1 = 1$. We do so by induction on n.

Theorem 6.1. *The sequence defined by*

$$H_n = 2H_{n-1} + 1, \quad H_1 = 1,$$

has the closed-form solution $H_n = 2^n - 1$ *for all integers* $n \geq 1$.

Proof. We proceed by induction on n.

Base case ($n = 1$).

$$2^1 - 1 = 1 = H_1.$$

Inductive step. Assume the formula holds for some integer $n \geq 1$, i.e.,

$$H_n = 2^n - 1. \tag{IH}$$

We wish to show that $H_{n+1} = 2^{n+1} - 1$.

By the recurrence relation (1),

$$H_{n+1} = 2H_n + 1.$$

Substituting the inductive hypothesis (IH):

$$H_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1,$$

which is precisely the formula evaluated at $n + 1$.

By the principle of mathematical induction, $H_n = 2^n - 1$ holds for all integers $n \geq 1$. □

Remark. The Tower of Hanoi puzzle with n disks requires exactly $2^n - 1$ moves — an exponential number of moves in the size of the input.