The Set of Real Numbers is Uncountable

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1 Cardinality and Isomorphism

Definition 1.1 (Isomorphism)**.** Two sets A and B are **isomorphic** (or **equinumerous**) if and only if there exists a bijection between them. In that case we write $A \sim B$.

Definition 1.2 (Same Cardinality)**.** Given two sets A and B, if $A \sim B$ we say that A and B are **equinumerous** or, equivalently, that they have the **same cardinality**.

Definition 1.3 (Finite Cardinality)**.** For every $n \in \mathbb{N}$ define the set

$$J_n = \begin{cases} \emptyset & \text{if } n = 0, \\ \{1, \ldots, n\} & \text{if } n > 0. \end{cases}$$

A set A has **finite cardinality** if and only if $\exists\, n \in \mathbb{N}$ such that $A \sim J_n$.

Definition 1.4 (Infinite Cardinality)**.** A set A has **infinite cardinality** if and only if $\nexists\, n \in \mathbb{N}$ such that $A \sim J_n$ (i.e. A does *not* have finite cardinality).

2 Countable and Uncountable Sets

Definition 2.1 (Countable Set)**.** An infinite set A is **countable** if and only if $A \sim \mathbb{N}$.

Example 2.2. The following infinite sets are countable.

- $\mathbb{N}$ itself: the identity function $\mathrm{id}\colon \mathbb{N} \to \mathbb{N}$, $\mathrm{id}(n) = n$, is trivially a bijection, so $\mathbb{N} \sim \mathbb{N}$.
- $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$: a bijection $f\colon \mathbb{N} \to \mathbb{N}^+$ is given by $f(n) = \mathrm{succ}(n)$.
- The set of even numbers $\mathbb{E}$: a bijection $f\colon \mathbb{N} \to \mathbb{E}$ is given by $f(n) = 2n$.

Definition 2.3 (Uncountable Set)**.** An infinite set A is **uncountable** if and only if $A \nsim \mathbb{N}$.

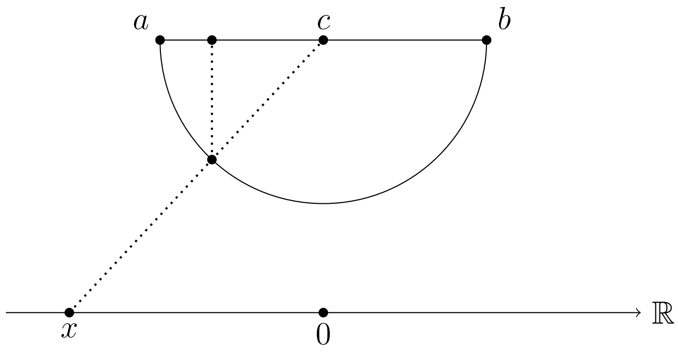
3 $\mathbb{R}$ is Uncountable

Theorem 3.1. $\mathbb{R}$ *is not countable, i.e.* $\mathbb{R} \nsim \mathbb{N}$.

Proof. The proof proceeds in three steps.

Step 1. $\mathbb{R} \sim (a, b)$ for all $a, b \in \mathbb{R}$ with $a < b$.

We show geometrically that there exists a bijection between any open interval (a, b) and $\mathbb{R}$. Place the segment $[a, b]$ horizontally above the real line. Draw the lower semicircle with diameter $[a, b]$ and centre c. Given any point $x \in \mathbb{R}$, draw the line from x to c; it meets the semicircle at a unique point, from which a perpendicular to $[a, b]$ yields a unique point in (a, b). This defines a bijection $\mathbb{R} \leftrightarrow (a, b)$; choosing $a = 0$ and $b = 1$ gives $\mathbb{R} \sim (0, 1)$.



How the bijection works in practice.

- **From $\mathbb{R}$ to (a, b):** pick any point x on the real line. Draw the straight line from x to the centre c; it crosses the semicircle at exactly one point. From that crossing point, drop a vertical line up to the segment $[a, b]$: the point where it lands is the image of x in (a, b).
- **From (a, b) to $\mathbb{R}$:** pick any point p in (a, b). Drop a vertical line from p down to the semicircle; it meets the arc at exactly one point. Draw the straight line from that arc point through c and extend it until it hits the real line: that intersection is the image of p in $\mathbb{R}$.

Remark 3.1. The diameter used to construct the semicircle is the closed segment $[a, b]$, yet the bijection targets the *open* interval (a, b). To see why the endpoints are excluded, recall how a point $x \in \mathbb{R}$ is mapped to a point in (a, b): one draws the line from x to c, finds where it meets the semicircle, and projects vertically up to (a, b).

Now imagine sliding x further and further to the left along $\mathbb{R}$. The line from x to c becomes increasingly flat — nearly horizontal — and its intersection with the semicircle drifts closer and closer to the leftmost point of the arc, which lies directly above a. In the limit, as $x \to -\infty$, the intersection would reach a itself; but $-\infty$ is not a real number, so a is never actually attained by any $x \in \mathbb{R}$. The same reasoning applies symmetrically: b would only be reached in the limit $x \to +\infty$. Since neither limit corresponds to any element of $\mathbb{R}$, the endpoints a and b have no pre-image, and the range of the map is precisely the open interval (a, b).

Step 2. $\mathbb{N} \not\sim (0, 1)$ (*by diagonalization*).

Suppose, for contradiction, that $(0, 1) \sim \mathbb{N}$. Under this hypothesis, $(0, 1)$ can be listed exhaustively. Consider then *any* exhaustive list of elements of $(0, 1)$:

$$\begin{array}{llllll} 0, & \underline{a_{00}} & a_{01} & a_{02} & a_{03} & \cdots \\ 0, & a_{10} & \underline{a_{11}} & a_{12} & a_{13} & \cdots \\ 0, & a_{20} & a_{21} & \underline{a_{22}} & a_{23} & \cdots \\ & \vdots & & & & \end{array}$$

where $a_{ij} \in \{0, 1, \ldots, 9\}$ is the j-th decimal digit of the i-th element in the list.

Construct the real number $0, d_0\, d_1\, d_2\, \dots$ by setting

$$d_i = \begin{cases} 2 & \text{if } a_{ii} \neq 2, \\ 3 & \text{if } a_{ii} = 2. \end{cases}$$

Since every digit d_i belongs to $\{2, 3\}$, the constructed number lies in $(0, 1)$. We claim, however, that it does *not* appear in the list:

$$\begin{aligned} d_0 \neq a_{00} &\implies 0, d_0 d_1 d_2 \dots \neq 0, a_{00}\, a_{01}\, a_{02}\, \dots \\ d_1 \neq a_{11} &\implies 0, d_0 d_1 d_2 \dots \neq 0, a_{10}\, a_{11}\, a_{12}\, \dots \\ d_2 \neq a_{22} &\implies 0, d_0 d_1 d_2 \dots \neq 0, a_{20}\, a_{21}\, a_{22}\, \dots \\ &\vdots \end{aligned}$$

The i-th digit of $0, d_0 d_1 d_2 \dots$ always differs from the i-th digit of the i-th element in the list, so the constructed number differs from *every* element in the list.

Therefore the list is **not exhaustive** — a contradiction.

Step 3. $\mathbb{R} \not\sim \mathbb{N}$.

By Step 1, $\mathbb{R} \sim (0, 1)$; by Step 2, $(0, 1) \not\sim \mathbb{N}$. By the transitivity of $\sim$ it follows that

$$\mathbb{R} \not\sim \mathbb{N},$$

i.e. $\mathbb{R}$ is uncountable. □

4 Cantor's Diagonal Argument in Simpler Terms

This section presents the diagonalization of Step 2 in simpler terms.

The assumption. Suppose one asserts that every real number in $(0, 1)$ can be placed in a one-to-one correspondence with the natural numbers, i.e. every element of (0,1) appears exactly once in the list.

$$\begin{aligned} 1 &\longleftrightarrow 0.12345\dots \\ 2 &\longleftrightarrow 0.52678\dots \\ 3 &\longleftrightarrow 0.79354\dots \\ 4 &\longleftrightarrow 0.63948\dots \\ 5 &\longleftrightarrow 0.72835\dots \\ &\vdots \end{aligned}$$

Suppose, in other words, that this enumeration is *exhaustive.*

The diagonal construction. We exhibit a number in $(0, 1)$ that does not appear anywhere in the list. Identify the *diagonal digits*: the 1st decimal digit of the 1st entry, the 2nd decimal digit of the 2nd entry, and in general the i-th decimal digit of the i-th entry. Construct a new number by, for example, increasing each diagonal digit by 1 (wrapping 9 around to 0 if necessary).

$$
\begin{array}{ccl}
1 & \longleftrightarrow & 0.\textcircled{1}2345\ldots \\
2 & \longleftrightarrow & 0.5\textcircled{2}678\ldots \\
3 & \longleftrightarrow & 0.79\textcircled{3}54\ldots \\
4 & \longleftrightarrow & 0.639\textcircled{4}8\ldots \\
5 & \longleftrightarrow & 0.7283\textcircled{5}\ldots \\
 & \vdots & \qquad\big\downarrow \\
\end{array}
$$

$$\textbf{new number:}\quad 0.23456\ldots$$

The diagonal digits are $1, 2, 3, 4, 5$; increasing each by 1 yields the new number $0.23456\ldots$, which belongs to $(0, 1)$.

Why the new number is absent from the list. By construction, the new number differs from the 1st entry in its 1st decimal digit, from the 2nd entry in its 2nd decimal digit, and more generally from the n-th entry in its n-th decimal digit. It therefore cannot coincide with any entry in the list, which contradicts the assumption that the enumeration was exhaustive.

The conclusion. Since this construction applies to *any* proposed enumeration of $(0, 1)$, no bijection between $\mathbb{N}$ and $(0, 1)$ can exist, and hence, by Step 1, no bijection between $\mathbb{N}$ and $\mathbb{R}$ can exist. The set of real numbers is therefore uncountable.

This argument is known as **Cantor's diagonal argument**, after Georg Cantor, who published it in 1891.